Available online at www.sciencedirect.com

ScienceDirect

Procedia Structural Integrity 3 (2017) 526–544

Structural Integrity Procedia

www.elsevier.com/locate/procedia

XXIV Italian Group of Fracture Conference, 1-3 March 2017, Urbino, Italy

Torsion problems of finite cylinders weakened by ring-shaped cracks

Nataly Vaysfeld*, Yuriy Protcerov

Odesa Mechnikov University, Dvoryans'ka str.2, Odesa, Ukraine

Abstract

A finite elastic hollow and solid cylinders are considered. The bottom faces of the cylinders are fixed, the upper faces are free from stress. The tangent axisymmetric loading is applied along their cylindrical surfaces. This leads to the torsion axisymmetric deformation. A system of N ring – shaped cracks is situated inside the cylinders parallel to the cylinder's axis. It is supposed that the branches of the cracks are free from stress. It is necessary to construct the formulas for the stress intensity factor calculation and investigate the stress state of a solid. The initial boundary value problem is reduced with Fourier transformation to a system of integral singular equations with regard to the jumps of the displacements at the cracks' branches. The singularity of the equations kernels is extracted. The system of singular integral equations is solved with the orthogonal polynomial method. The solution of the system is searched as the series by Chebyshev polynomials with the weight function. The realisation of orthogonal polynomial method leads to an infinite system of linear algebraic equations with regard to the unknown coefficients of the series. The formulas for the stresses and displacements of an elastic finite cylinder are presented. The numerical realisation of the proposed method is demonstrated in cases with two and three cracks ; the stress state is investigated dependingon the cracks' locations and sizes.

Peer-review under responsibility of the Scientific Committee of IGF Ex-Co.

Keywords: Finite cylinder; ring –shaped crack; orthogonal polynomials method; stress intensity factor

1. Introduction

Elastic cylinders of finite length (both solid and hollow) are often used as components of machines and in building construction. The influence of a crack may impair the correct function and even to total destruction of such components and result in failure of the machine or construction. It is necessary to establish a corresponding

* Corresponding author. Tel.: (380 48) 723-49-60.
E-mail address: vaysfeld@onu.edu.ua

Copyright © 2017 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Peer-review under responsibility of the Scientific Committee of IGF Ex-Co.
10.1016/j.prostr.2017.04.046

mathematical model in order to represent the impact of various different parameters on the system, to state the most dangerous location of a crack and to estimate the most high stress inside a body.

A lot of work is dedicated to the investigation of bodies with defects (cracks and inclusions Morozov (1984), Savruk et al (1989), Panasyk et al (1981), Aleksandrov et al (1993), Babeshko et al (2007), Hakobyan (2014), Lee (2004), Mykhas'kiv et al (2009), Chang et al (2014), Xie et al (2003), Jin-Chad et al (1996). The stress state and stress intensity factor of homogeneous and layered cylinders with circular cracks is investigated by many authors Chang (1985), Yantian et al (1988), Zhang (1988), Akiyawa et al (2001), Huang et al (2005), Kaman et al (2006). The idea of the solving methods is based on a problem's reductum to a system of singular equations of Cauchy type or to Fredholm's type equation, solved numerically. In Protserov and Vaysfeld (2017) the problem results in a system of integro-differential equations, solved by the orthogonal polynomial method. The arc crack is considered in Gribova et al (1989), where the problem is reduced to the Riemann problem. The torsion problems of solid, hollow and two layered cylinders with cylindrical (interface) cracks are solved in Wuthrich (1980), Yong et al (2013), Shi (2015), Pengpeng (2015).

Less work has studied ring-shaped cracks. The torsion problem solutions for a cylinder with external ring-shaped cracks are represented in Kudryavcev et al (1973), Malits (2009). In Suzuki et al (1980) the solution is constructed for a ring-shaped crack on the internal surface of a hollow cylinder. But there are fewer papers where authors solve the problems for cylinders with the internal ring-shaped cracks. At Aleksandrov et al (1993), Kanwal (1974) the problem with the ring-shaped crack is solved for the unbounded medium. Only in Han et al (1994) the problem with one internal ring-shaped crack is considered for the case of cylinder torsion. So the problem of stress state estimation during the torsion of the cylinders weakened by the internal ring-shaped cracks needs further investigation and study.

Nomenclature

R	external radius of cylinder
H	height of cylinder
G	share modulus
u	tangential displacement
K_{iii}	stress intensity factor (SIF)

1. Problem's statement

Let's consider a solid (the problem №1) and a hollow (the problem №2) elastic finite cylinders occupying areas in the cylindrical coordinate system (r,φ,z) $0 \le r \le R, -\pi \le \varphi \le \pi, 0 \le z \le H$ $R_0 \le r \le R, -\pi \le \varphi \le \pi, 0 \le z \le H$ correspondently. The lower bases of the cylinders are fixed, upper bases are free from stresses. The axisymmetric torsion loading is applied to the lateral surface $r = R$ of the cylinders. This loading causes the torsion of the solids. In the case of a hollow cylinder it is supposed that internal cylindrical surface $r = R_0$ is free from stress. The system of N ring-shaped cracks is situated inside the cylinders on the segments $z = d_j, a_j \le r \le b_j, j = \overline{1,N}$, the branches of the cracks are free from stress. The axisymmetric statement of the problems leads to the only one nonzero displacement $u_\varphi(r,z)$, satisfying the torsion equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_\varphi}{\partial r}\right) - \frac{1}{r}u_\varphi + r\frac{\partial^2 u_\varphi}{\partial z^2} = 0.$$

The only nonzero stress are the tangential stress

$$\tau_{r\varphi} = G\left(\frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi\right), \quad \tau_{z\varphi} = G\frac{\partial u_\varphi}{\partial z},$$

The boundary conditions for the solid cylinder (problem №1) are written in the form

$$u_\varphi(r,0) = 0,\ \tau_{z\varphi}(r,H) = G\frac{\partial u_\varphi}{\partial z}|_{z=H} = 0\ ,\ \tau_{r\varphi} = G\left(\frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi\right)$$

The boundary conditions for the hollow cylinder (problem №2) have the same presentations but are supplemented with the condition on the internal cylindrical surface:

$$\tau_{r\varphi}(R_0,z) = G\left(\frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi\right)_{r=R_0} = 0$$

Displacements is discontinuous on the cracks' surfaces

$$u_\varphi(r,d_j-0) - u_\varphi(r,d_j+0) = \omega_j(r),\ \tau_{z\varphi}(r,d_j\pm 0) = 0,\ a_j \le r \le b_j,\ j=\overline{1,N},$$

here $\omega_j(r)$ are the unknown jumps of the displacements on the cracks' branches, $\omega_j(r) = 0$ outside the segment of cracks' location.

Let's pass to the dimensionless coordinates $\rho = rR^{-1}, \zeta = zH^{-1}$ and designate

$$u(\rho,\zeta) = u_\varphi(R\rho,H\zeta),\ p(\zeta) = q(H\zeta),\ \chi_j(\rho) = \omega_j(R\rho),\ \gamma = RH^{-1},\ \varepsilon = R_0R^{-1},\ \delta_j = d_jH^{-1},$$
$$\alpha_j = a_jR^{-1},\ \beta_j = b_jR^{-1},\ j=\overline{1,N}.$$

One must find the solution of the equation

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) - \frac{1}{\rho}u + \gamma^2\rho\frac{\partial^2 u}{\partial\zeta^2} = 0,\ 0 \le \rho < 1\ (\varepsilon < \rho < 1),\ 0 < \zeta < 1 \tag{1}$$

with the boundary conditions

$$u(\rho,0) = 0,\ \frac{\partial u}{\partial\zeta}|_{\zeta=1} = 0 \tag{2}$$

$$\left(\frac{\partial u}{\partial\rho} - \frac{1}{\rho}u\right)_{\rho=1} = RG^{-1}p(\zeta) \tag{3}$$

$$\left(\frac{\partial u}{\partial\rho} - \frac{1}{\rho}u\right)_{\rho=1} = RG^{-1}p(\zeta),\ \left(\frac{\partial u}{\partial\rho} - \frac{1}{\rho}u\right)_{\rho=\varepsilon} = 0 \tag{4}$$

Boundary conditions (2), (3) should be satisfied for the problem №1; boundary conditions (2) – (4) should be satisfied for the problem №2. The conditions on the cracks' surfaces should also be satisfied

$$u\left(\rho,\delta_j-0\right)-u\left(\rho,\delta_j+0\right)=\chi_j\left(\rho\right),\ j=\overline{1,N} \tag{5}$$

$$\frac{\partial u}{\partial\zeta}\Big|_{\delta_j\pm 0}=0,\ j=\overline{1,N} \tag{6}$$

2. The construction of one-dimensional problem and its solution

To get the one-dimensional boundary value problem one must use finite integral Fourier transformation with regard to variable ζ

$$u_k\left(\rho\right)=\int_0^1 u\left(\rho,\zeta\right)\sin\lambda_k\zeta\,d\zeta,\ u\left(\rho,\zeta\right)=2\sum_{k=1}^{\infty}u_k\left(\rho\right)\sin\lambda_k\zeta,\ \lambda_k=\frac{\pi}{2}\left(2k-1\right). \tag{7}$$

The application of transformation (7) to the equation(1)and boundary conditions (3), (4) (the boundary conditions(2) will be satisfied during it) accordingly to the generalized scheme Popov (1982) with regard to the conditions on the cracks (5) leads to the one dimensional boundary value problem

$$\left(\rho u_k'\left(\rho\right)\right)'-\left(\rho^{-1}+\gamma^2\lambda_k^2\rho\right)u_k\left(\rho\right)=\gamma^2\lambda_k\rho\sum_{j=1}^{N}\cos\lambda_k\delta_j\chi_j\left(\rho\right),\ 0<\rho<1\left(\varepsilon<\rho<1\right) \tag{8}$$

$$u_k'\left(1\right)-u_k\left(1\right)=RG^{-1}p_k,\ p_k=\int_0^1 p\left(\zeta\right)\sin\lambda_k\zeta\,d\zeta \tag{9}$$

$$u_k'\left(1\right)-u_k\left(1\right)=RG^{-1}p_k,\ \varepsilon u_k'\left(\varepsilon\right)-u_k\left(\varepsilon\right)=0 \tag{10}$$

The general solution of the equation (1) for the problem №1 has the form

$$u_k\left(\rho\right)=A_kI_1\left(\gamma\lambda_k\rho\right)+\chi^2\lambda_k\sum_{k=1}^{N}\cos\lambda_k\delta_j\int_{\alpha_j}^{\beta_j}\Phi_k\left(\rho,t\right)\chi_j\left(t\right)t\,dt, \tag{11}$$

For the Problem №2 the general solution is represented by the formula

$$u_k\left(\rho\right)=A_kI_1\left(\gamma\lambda_k\rho\right)+B_kK_1\left(\gamma\lambda_k\rho\right)+\chi^2\lambda_k\sum_{k=1}^{N}\cos\lambda_k\delta_j\int_{\alpha_j}^{\beta_j}\Phi_k\left(\rho,t\right)\chi_j\left(t\right)t\,dt, \tag{12}$$

where $I_1\left(x\right),K_1\left(x\right)$ are modified Bessel functions of first order, A_k,B_k are the integration constants. The fundamental function Kamke (1976) of equation (8) has the form

$$\Phi_k\left(\rho,t\right)=-\begin{cases}K_1\left(\gamma\lambda_k\rho\right)I_1\left(\gamma\lambda_kt\right),\ 0\le t<\rho\le 1\\ I_1\left(\gamma\lambda_k\rho\right)K_1\left(\gamma\lambda_kt\right),\ 0\le\rho<t\le 1\end{cases}$$

it is easy to check that this function satisfies the homogeneous equation (9) , is bounded when $\rho=0,$ on each of the segments $\left[0;t\right),\left(t;1\right],$ and is continuous.

Its derivative $\dfrac{\partial\Phi_k\left(\rho,t\right)}{\partial\rho}$ has the jump $\dfrac{1}{t},$ during the passing across line $\rho=t.$ It is possible to represent fundamental function (12) in another form with the help of the formula (6.541, Gradshtein et al (1963))

$$\Phi_k(\rho,t) = -\int_0^\infty J_1(\rho x) J_1(tx) \frac{dx}{x^2+\gamma^2\lambda_k^2}, \tag{13}$$

$$\Phi_k(\rho,t) = -\int_0^\infty J_1(\rho x) J_1(tx) \frac{dx}{x^2+\gamma^2\lambda_k^2}, \tag{14}$$

where $J_1(x)$ is Bessel function.

For the problem №1 with regard to the correspondences $xI_1'(x) - I_1(x) = xI_2(x),\ xK_1'(x) - K_1(x) = -xK_2(x)$ from the boundary conditions (9) one finds $A_k = \dfrac{Hp_k}{G\lambda_k I_2(\gamma\lambda_k)} - \gamma^2\lambda_k \dfrac{K_2(\gamma\lambda_k)}{I_2(\gamma\lambda_k)} \sum_{j=1}^N \cos\lambda_k\delta_j \int_{\alpha_j}^{\beta_j} I_1(\gamma\lambda_k t)\chi_j(t)t\,dt.$

For the problem №1 from the boundary conditions (10) one finds

$$A_k = \frac{Hp_k K_2(\gamma\lambda_k\varepsilon)}{G\lambda_k\Delta(k)} - \gamma^2\lambda_k \frac{K_2(\gamma\lambda_k)}{\Delta(k)} \sum_{j=1}^N \cos\lambda_k\delta_j \int_{\alpha_j}^{\beta_j} \left[I_2(\gamma\lambda_k\varepsilon)K_1(\gamma\lambda_k t) + K_2(\gamma\lambda_k\varepsilon)I_1(\gamma\lambda_k t)\right]\chi_j(t)t\,dt$$

$$B_k = \frac{Hp_k I_2(\gamma\lambda_k\varepsilon)}{G\lambda_k\Delta(k)} - \gamma^2\lambda_k \frac{I_2(\gamma\lambda_k\varepsilon)}{\Delta(k)} \sum_{j=1}^N \cos\lambda_k\delta_j \int_{\alpha_j}^{\beta_j} \left[I_2(\gamma\lambda_k)K_1(\gamma\lambda_k t) + K_2(\gamma\lambda_k)I_1(\gamma\lambda_k t)\right]\chi_j(t)t\,dt$$

where $\Delta(k) = I_2(\gamma\lambda_k)K_2(\gamma\lambda_k\varepsilon) - I_2(\gamma\lambda_k\varepsilon)K_2(\gamma\lambda_k)$.

One should substitute the found values of the integration constants in the corresponding equalities (11) and (12) and use the inversion formula (7). As a result, the expressions of the displacement will be constructed

for Problem №1

$$u(\rho,\zeta) = \frac{2H}{G}\sum_{k=1}^\infty p_k \frac{I_1(\gamma\lambda_k\rho)}{\lambda_k I_2(\gamma\lambda_k)} \sin\lambda_k\zeta - \frac{1}{2}\sum_{j=1}^N \int_{\alpha_j}^{\beta_j} \chi_j(t)t\,dt \int_0^\infty J_1(\rho x)J_1(tx)\left[ch\frac{x}{\gamma}\left(1-\zeta-\delta_j\right) + \right.$$
$$\left. + \operatorname{sgn}(\zeta-\delta_j)\, ch\frac{x}{\gamma}\left(1-\left|\zeta-\delta_j\right|\right)\right] \operatorname{sech}\frac{x}{\gamma}\cdot x dx - 2\gamma^2 \sum_{j=1}^N \int_{\alpha_j}^{\beta_j} \left[\sum_{k=1}^\infty \lambda_k \cos\lambda_k\delta_j \sin\lambda_k\zeta \frac{K_2(\gamma\lambda_k)}{I_2(\gamma\lambda_k)} \cdot \right.$$
$$\left. \cdot I_1(\gamma\lambda_k\rho) I_1(\gamma\lambda_k t)\right]\chi_j(t)t\,dt \tag{15}$$

for Problem №2 $$u(\rho,\zeta) = \frac{2H}{G}\sum_{k=1}^\infty p_k \frac{F_k(\rho)}{\lambda_k\Delta(k)} \sin\lambda_k\zeta - \frac{1}{2}\sum_{j=1}^N \int_{\alpha_j}^{\beta_j} \chi_j(t)t\,dt \int_0^\infty J_1(\rho x)J_1(tx)\left[ch\frac{x}{\gamma}\left(1-\zeta-\delta_j\right) + \right. \tag{16}$$

$$\left. + \operatorname{sgn}(\zeta-\delta)\, ch\frac{x}{\gamma}\left(1-\left|\zeta-\delta_j\right|\right)\right] \operatorname{sech}\frac{x}{\gamma}\cdot x dx - 2\gamma^2 \sum_{j=1}^N \int_{\alpha_j}^{\beta_j} \left[\sum_{k=1}^\infty \lambda_k \cos\lambda_k\delta_j \sin\lambda_k\zeta \frac{N_k(\rho,t)}{\Delta(k)}\right]\chi_j(t)t\,dt.$$

Here $F_k(\rho) = K_2(\gamma\lambda_k\varepsilon)I_1(\gamma\lambda_k\rho) + I_2(\gamma\lambda_k\varepsilon)K_1(\gamma\lambda_k\rho)$

$N_k(\rho,t) = K_2(\gamma\lambda_k)I_1(\gamma\lambda_k\rho)\left[I_2(\gamma\lambda_k\varepsilon)K_1(\gamma\lambda_k t) + K_2(\gamma\lambda_k\varepsilon)I_1(\gamma\lambda_k t)\right] +$

$$+I_2\left(\gamma\lambda_k\varepsilon\right)K_1\left(\gamma\lambda_k\rho\right)\left[I_2\left(\gamma\lambda_k\right)K_1\left(\gamma\lambda_k t\right)+K_2\left(\gamma\lambda_k\right)I_1\left(\gamma\lambda_k t\right)\right].$$

The expression (14) was used during formulas (15), (16) construction. The formula (1.445(1), Gradshtein et al (1963)) was used to find the series

$$\sum_{k=1}^{\infty}\frac{\lambda_k\cos\lambda_k\delta_j\sin\lambda_k\zeta}{x^2+\gamma^2\lambda_k^2}=\frac{1}{4\gamma^2}\left[\operatorname{sgn}\left(\zeta-\delta_j\right)ch\frac{x}{\gamma}\left(1-\left|\zeta-\delta_j\right|\right)+ch\frac{x}{\gamma}\left(1-\zeta-\delta_j\right)\right]\operatorname{sech}\frac{x}{\gamma}.$$

4. Obtaining the integral equation system and its solution with the orthogonal polynomials method

The unknown functions $\chi_j\left(\rho\right)$ (the jump of stress through the branches of j -d crack) are the components of the displacement formulas (15) and (16). To find them one must use the conditions (6) and to demand the absence of the stress on cracks' branches.

Let's start with Problem №2. The expression (15) should be substituted in condition (6) for a crack

$$\frac{\partial u}{\partial\zeta}|_{\zeta=\delta_i}=\frac{2H}{G}\sum_{k=1}^{\infty}p_k\frac{I_1\left(\gamma\lambda_k\rho\right)}{I_2\left(\gamma\lambda_k\right)}\cos\lambda_k\delta_i+\frac{1}{2\gamma}\sum_{j=1}^{N}\int_{\alpha_j}^{\beta_j}\chi_j\left(t\right)t\,dt\int_0^{\infty}J_1\left(\rho x\right)J_1\left(t\,x\right)\left[sh\frac{x}{\gamma}\left(1-\delta_i-\delta_j\right)+\right.$$

$$\left.+sh\frac{x}{\gamma}\left(1-\left|\delta_i-\delta_j\right|\right)\right]x^2\operatorname{sech}\frac{x}{\gamma}dx-2\gamma^2\sum_{j=1}^{N}\int_{\alpha_j}^{\beta_j}\left[\sum_{k=1}^{\infty}\lambda_k^2\cos\lambda_k\delta_i\cos\lambda_k\delta_j\frac{K_2\left(\gamma\lambda_k\right)}{I_2\left(\gamma\lambda_k\right)}\cdot\right.$$

$$\left.\cdot I_1\left(\gamma\lambda_k\rho\right)I_1\left(\gamma\lambda_k t\right)\right]\chi_j\left(t\right)t\,dt=0,\ i=\overline{1,N}.$$

The identity $x^2J_1\left(\rho x\right)J_1\left(tx\right)=\frac{\partial^2}{\partial\rho\partial t}J_0\left(\rho x\right)J_0\left(tx\right)$ Gradshtein et al (1963) is used for the extraction of singular kernel. It leads to the system of the integral equations

$$\frac{d}{d\rho}\int_{\alpha_i}^{\beta_i}\frac{\partial}{\partial t}W_{00}^0\left(\rho,t\right)\chi_i\left(t\right)t\,dt+\sum_{j=1}^{N}\int_{\alpha_j}^{\beta_j}\left[S_{ij}\left(\rho,t\right)-R_{ij}\left(\rho,t\right)\right]\chi_j\left(t\right)t\,dt=G^{-1}f_i\left(\rho\right),\tag{17}$$

$$\alpha_i<\rho<\beta_i,\ i=\overline{1,N}$$

where $W_{\lambda\mu}^{\nu}\left(\rho,t\right)=\int_0^{\infty}J_\lambda\left(\rho x\right)J_\mu\left(t\,x\right)x^{\nu}dx$ is Weber-Schafheitlin discontinuous integral Beitman et al (1974), with singularity when $\rho=t$,

$$S_{ij}\left(\rho,t\right)=\int_0^{\infty}J_1\left(\rho x\right)J_1\left(t\,x\right)\left[sh\frac{x}{\gamma}\left(1-\delta_i-\delta_j\right)+sh\frac{x}{\gamma}\left(1-\left|\delta_i-\delta_j\right|\right)\right]x^2\operatorname{sech}\frac{x}{\gamma}dx,\ i\neq j$$

$$S_{ii}\left(\rho,t\right)=\int_0^{\infty}J_1\left(\rho x\right)J_1\left(t\,x\right)\left[sh\frac{x}{\gamma}\left(1-2\delta_i\right)\operatorname{sech}\frac{x}{\gamma}+th\frac{x}{\gamma}-1\right]x^2dx$$

$$R_{ij}\left(\rho,t\right)=4\gamma^3\sum_{k=1}^{\infty}\lambda_k^2\cos\lambda_k\delta_i\cos\lambda_k\delta_j\frac{K_2\left(\gamma\lambda_k\right)}{I_2\left(\gamma\lambda_k\right)}I_1\left(\gamma\lambda_k\rho\right)I_1\left(\gamma\lambda_k t\right)$$

are the regular kernels, $f_i(\rho) = -4R\sum_{k=1}^{\infty} p_k \frac{I_1(\gamma\lambda_k\rho)}{I_2(\gamma\lambda_k)}\cos\lambda_k\delta_i$.

All integrals in system (17) should be integrated by parts with regard to a crack's closeness condition $\chi_j(\alpha_j) = \chi_j(\beta_j) = 0$, then all equations of this system should be integrated with regard to variable ρ. One must use that $W_{00}^0(\rho,t) = \frac{2}{\pi(\rho+t)}K\left(\frac{2\sqrt{\rho t}}{\rho+t}\right)$, where $K(x)$ is full elliptical integral of 1-st order (obtaining this formula is shown at App. A). After these transformations the system (17) takes the form

$$\frac{2}{\pi}\int_{\alpha_i}^{\beta_i} K\left(\frac{2\sqrt{\rho t}}{\rho+t}\right)\frac{\psi_i(t)}{\rho+t}dt + \sum_{j=1}^{N}\int_{\alpha_j}^{\beta_j}\left[S_{ij}^*(\rho,t) - R_{ij}^*(\rho,t)\right]\psi_j(t)dt = f_i^*(\rho) + C_i, \alpha_i < \rho < \beta_i, i = \overline{1,N} \tag{18}$$

where $\psi_i(t) = G^{-1}\frac{d}{dt}\left[t\chi_i(t)\right]$, $f_i^*(\rho) = 4H\sum_{k=1}^{\infty} p_k \frac{I_0(\gamma\lambda_k\rho)}{\lambda_k I_2(\gamma\lambda_k)}\cos\lambda_k\delta_i$,

$$S_{ij}^*(\rho,t) = \int_0^\infty J_0(\rho x)J_0(tx)\left[sh\frac{x}{\gamma}\left(1-\delta_i-\delta_j\right) + sh\frac{x}{\gamma}\left(1-\left|\delta_i-\delta_j\right|\right)\right]\text{sech}\frac{x}{\gamma}dx, \, i \neq j,$$

$$S_{ii}^*(\rho,t) = \int_0^\infty J_0(\rho x)J_0(tx)\left[sh\frac{x}{\gamma}(1-2\delta_i)\text{sech}\frac{x}{\gamma} + th\frac{x}{\gamma} - 1\right]dx,$$

$$R_{ij}^*(\rho,t) = 4\gamma\sum_{k=1}^{\infty}\cos\lambda_k\delta_i\cos\lambda_k\delta_j\frac{K_2(\gamma\lambda_k)}{I_2(\gamma\lambda_k)}I_0(\gamma\lambda_k\rho)I_0(\gamma\lambda_k t),$$

C_i are the unknown constants.

The transition to the new variables was done in the obtained system of integral equations

$$\rho = \alpha_i\exp\left(\frac{1+\xi}{\mu_i}\right), t = \alpha_i\exp\left(\frac{1+\eta}{\mu_i}\right), \mu_i = 2\left(\ln\frac{\beta_i}{\alpha_i}\right)^{-1},$$

It reduces the interval of integration $(\alpha_i;\beta_i)$ to the interval $(-1;1)$. After the transition the resulting system of the integral equations will be following:

$$\int_{-1}^{1}\text{sech}\frac{\xi-\eta}{2\mu_i}K\left(\text{sech}\frac{\xi-\eta}{2\mu_i}\right)\theta_i(\eta)d\eta + \sum_{j=1}^{N}\int_{-1}^{1}\left[M_{ij}(\xi,\eta) - L_{ij}(\xi,\eta)\right]\theta_j(\eta)d\eta = g_i(\xi) + C_i h_i(\xi)$$
$$-1 < \xi < 1, i = \overline{1,N} \tag{19}$$

where $\theta_i(\eta) = \sqrt{\alpha_i}\exp\left(\frac{1+\eta}{2\mu_i}\right)\psi_i\left(\alpha_i\exp\left(\frac{1+\eta}{\mu_i}\right)\right)$,

$$M_{ij}(\xi,\eta) = \pi\frac{\mu_i}{\mu_j}\sqrt{\alpha_i\alpha_j}\exp\left(\frac{1+\xi}{2\mu_i} + \frac{1+\eta}{2\mu_j}\right)S_{ij}^*\left(\alpha_i\exp\left(\frac{1+\xi}{\mu_i}\right), \alpha_j\exp\left(\frac{1+\eta}{\mu_j}\right)\right),$$

$$L_{ij}(\xi,\eta)=4\pi\gamma\frac{\mu_i}{\mu_j}\sqrt{\alpha_i\alpha_j}\exp\left(\frac{1+\xi}{2\mu_i}+\frac{1+\eta}{2\mu_j}\right)R_{ij}^*\left(\alpha_i\exp\left(\frac{1+\xi}{\mu_i}\right),\alpha_j\exp\left(\frac{1+\eta}{\mu_j}\right)\right),$$

$$g_i(\xi)=4\pi H\mu_i\sqrt{\alpha_i}\exp\left(\frac{1+\xi}{2\mu_i}\right)f_i^*\left(\alpha_i\exp\left(\frac{1+\xi}{\mu_i}\right)\right),\ h_i(\xi)=\pi\mu_i\sqrt{\alpha_i}\exp\left(\frac{1+\xi}{2\mu_i}\right).$$

It is known from general theory of full elliptical integrals, that integral $K(x)$ has a logarithmic singularity at $x=1$. The singular kernels of system (19) with regard of it are represented as

$$\operatorname{sech}\frac{\xi-\eta}{2\mu_i}K\left(\operatorname{sech}\frac{\xi-\eta}{2\mu_i}\right)=\ln\frac{1}{|\xi-\eta|}+l_i(\xi-\eta),$$

where the functions $l_i(x)$ are even, continuous with their derivative and $\lim_{x\to0}l_i(x)=\ln 8\mu_i$.

The system of integral equations for the estimation of functions $\theta_i(\xi)$ is written finally

$$\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\theta_i(\eta)d\eta+\sum_{j=1}^{N}\int_{-1}^{1}\left[l_i(\xi-\eta)+M_{ij}(\xi,\eta)-L_{ij}(\xi,\eta)\right]\theta_j(\eta)d\eta=g_i(\xi)+C_ih_i(\xi), \tag{20}$$

$$-1<\xi<1,\ i=\overline{1,N}$$

All transformations for the Problem №2 should be done in an analogical way. These transformations lead at first to the system(18) where in the formulas for $f_i^*(\rho)$ ratio $\dfrac{I_0(\gamma\lambda_k\rho)}{I_2(\gamma\lambda_k)}$ should be changed to $\Delta(k)^{-1}\left[I_0(\gamma\lambda_k\rho)K_2(\gamma\lambda_k\varepsilon)-K_0(\gamma\lambda_k\rho)I_2(\gamma\lambda_k\varepsilon)\right]$, and in the formula for $R_{ij}^*(\rho,t)$ ratio $\dfrac{K_2(\gamma\lambda_k)}{I_2(\gamma\lambda_k)}I_0(\gamma\lambda_k\rho)I_0(\gamma\lambda_k t)$ should be changed too by the expression

$$\frac{I_2(\gamma\lambda_k\varepsilon)}{\Delta(k)}K_0(\gamma\lambda_k t)\left[K_0(\gamma\lambda_k\rho)I_2(\gamma\lambda_k)-I_0(\gamma\lambda_k\rho)K_2(\gamma\lambda_k)\right]-$$

$$-\frac{K_2(\gamma\lambda_k)}{\Delta(k)}I_0(\gamma\lambda_k t)\left[K_0(\gamma\lambda_k\rho)I_2(\gamma\lambda_k\varepsilon)-I_0(\gamma\lambda_k\rho)K_2(\gamma\lambda_k\varepsilon)\right].$$

The system of integral equations of type (20) is constructed after changing the variables. So, both problems lead to the system of the integral equations of type (20).

The structure of equation singular kernels in system (20) and availability of the spectral correspondence Popov (1982)

$$\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta=\sigma_nT_n(\xi),\ \sigma_n=\begin{cases}\pi\ln2,\ n=0\\ \dfrac{\pi}{n},\ n\ge1\end{cases}$$

allow the use of the orthogonal polynomial method to solve this system. Accordingly to the scheme of the method the solution of the system is searched as the series expansion by Chebyshev polynomials of I-st order $T_n(x)$

$$\theta_i(\eta)=\sum_{n=0}^{\infty}\theta_{in}\frac{T_n(\eta)}{\sqrt{1-\eta^2}},\ i=\overline{1,N}. \tag{21}$$

Realisation of the standard scheme of orthogonal polynomial methods leads to a system of linear algebraic equations with regard to the coefficients of the expansion (21)

$$k_m\theta_{im}+\sum_{j=1}^{N}\sum_{n=0}^{\infty}\left[A_{mn}^{i}+B_{mn}^{ij}-C_{mn}^{ij}\right]\theta_{jn}=g_{im}+C_i h_{im},\ i=\overline{1,N},\ m=0,1,2,... \tag{22}$$

where $k_m=\begin{cases}\pi^2\ln 2,\ m=0\\ \pi^2(2m)^{-1},\ m\ge 1\end{cases}$, $\quad A_{mn}^{i}=\int_{-1}^{1}T_m(\xi)\frac{d\xi}{\sqrt{1-\xi^2}}\int_{-1}^{1}l_i(\xi-\eta)\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta,$

$$B_{mn}^{ij}=\int_{-1}^{1}T_m(\xi)\frac{d\xi}{\sqrt{1-\xi^2}}\int_{-1}^{1}M_{ij}(\xi,\eta)\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta,\quad C_{mn}^{ij}=\int_{-1}^{1}T_m(\xi)\frac{d\xi}{\sqrt{1-\xi^2}}\int_{-1}^{1}L_{ij}(\xi,\eta)\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta,$$

$$g_{im}=\int_{-1}^{1}g_i(\xi)\frac{T_m(\xi)}{\sqrt{1-\xi^2}}d\xi,\ h_{im}=\int_{-1}^{1}h_i(\xi)\frac{T_m(\xi)}{\sqrt{1-\xi^2}}d\xi.$$

Right hand parts of the system (22) have the unknown constants C_i, so why one can search the solution of the system as the linear combinations

$$\theta_{im}=\theta_{im}^{0}+\sum_{s=1}^{N}C_s\theta_{im}^{s},\ i=\overline{1,N},\ m=0,1,2,... \tag{23}$$

where θ_{im}^{0} are the solutions of the system $k_m\theta_{im}^{0}+\sum_{j=1}^{N}\sum_{n=0}^{\infty}\left[A_{mn}^{i}+B_{mn}^{ij}-C_{mn}^{ij}\right]\theta_{jn}^{0}=g_{im}$, θ_{im}^{s} are the solutions of the system $k_m\theta_{im}^{s}+\sum_{j=1}^{N}\sum_{n=0}^{\infty}\left[A_{mn}^{i}+B_{mn}^{ij}-C_{mn}^{ij}\right]\theta_{jn}^{s}=h_{im}\delta_{is},\ s=\overline{1,N},\ \delta_{is}=\begin{cases}1,\ i=s\\ 0,\ i\ne s\end{cases}.$

After solving these systems and calculating the coefficients θ_{im}^{0} and θ_{im}^{s} one must find the constants C_i. Such grounds can be used with this aim. The functions $\psi_i(t)$, should be found from the system of the integral equations(18) and should satisfy the conditions

$$\int_{\alpha_i}^{\beta_i}\psi_i(t)dt=G^{-1}\int_{\alpha_i}^{\beta_i}\frac{d}{dt}\left[t\chi_i(t)\right]dt=G^{-1}t\chi_i(t)\Big|_{\alpha_i}^{\beta_i}=0,$$

which arise from the closeness condition of the crack $\chi_i(\alpha_i)=\chi_i(\beta_i)=0$. These conditions with regard to the executed change of the variables are equivalent to the next ones

$$\int_{-1}^{1}\exp\left(\frac{\eta}{2\mu_i}\right)\theta_i(\eta)d\eta=\sum_{n=0}^{\infty}\theta_{in}\int_{-1}^{1}\exp\left(\frac{\eta}{2\mu_i}\right)\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta=0,\ i=\overline{1,N},$$

Taking into consideration formula (2.18.1.10 , Prudnikov et al (1983)) and representation (23) one gets the system of linear equation to find the constants C_i

$$\sum_{s=1}^{N}C_s\sum_{n=0}^{\infty}\theta_{in}^{s}I_n\left(\frac{1}{2\mu_i}\right)=-\sum_{n=0}^{\infty}\theta_{in}^{0}I_n\left(\frac{1}{2\mu_i}\right),\ i=\overline{1,N}.$$

Thus, all values constituent in the formulas for the displacements (15) and (16) are found, from where it is not complicated to get the stress values inside the cylinders.

5. Finding the stress intensity factors (SIF)

SIF values are extremely interesting when solving the problems for solids with cracks. For the proposed problem such SIF values it is important to calculate on the internal and external contours of a crack . $K_{III}^{a_j}=\lim_{r\to a_j-0}\sqrt{2\pi\left(a_j-r\right)}\tau_{z\varphi}\left(r,d_j\right)$ and $K_{III}^{b_j}=\lim_{r\to b_j+0}\sqrt{2\pi\left(r-b_j\right)}\tau_{z\varphi}\left(r,d_j\right)$, $j=\overline{1,N}$.

With regard to all the executed earlier changeof the variables, integration by parts and truncation of summands having the finite limits when $r\to a_j-0$ and $r\to b_j+0$, one gets

$$
\begin{aligned}
K_{III}^{a_j}&=\frac{1}{\alpha_j\sqrt{2\pi R}}\lim_{\xi\to-1-0}\sqrt{1-\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi+1}{2}}}\cdot\frac{d}{d\xi}\left[\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\theta_j(\eta)d\eta\right]=\\
&=\frac{1}{\alpha_j\sqrt{2\pi R}}\sum_{n=0}^{\infty}\theta_{jn}\lim_{\xi\to-1-0}\sqrt{1-\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi+1}{2}}}\cdot\frac{d}{d\xi}\left[\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right]=\\
&=\frac{1}{\alpha_j\sqrt{2\pi R}}\sum_{n=0}^{\infty}\theta_{jn}\lim_{\xi\to-1-0}\sqrt{1-\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi+1}{2}}}\cdot\left[-\frac{1}{2\mu_j}\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta+\right.\\
&\qquad\left.+\exp\left(-\frac{1+\xi}{2\mu_j}\right)\frac{d}{d\xi}\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right].
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
K_{III}^{b_j}&=-\frac{1}{\sqrt{2\pi R\alpha_j\beta_j}}\lim_{\xi\to1+0}\sqrt{\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi-1}{2}}-1}\cdot\frac{d}{d\xi}\left[\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\theta_j(\eta)d\eta\right]=\\
&=-\frac{1}{\sqrt{2\pi R\alpha_j\beta_j}}\sum_{n=0}^{\infty}\theta_{jn}\lim_{\xi\to1+0}\sqrt{\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi-1}{2}}-1}\cdot\frac{d}{d\xi}\left[\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right]=\\
&=-\frac{1}{\sqrt{2\pi R\alpha_j\beta_j}}\sum_{n=0}^{\infty}\theta_{jn}\lim_{\xi\to1+0}\sqrt{\left(\beta_j\alpha_j^{-1}\right)^{\frac{\xi-1}{2}}-1}\cdot\left[-\frac{1}{2\mu_j}\exp\left(-\frac{1+\xi}{2\mu_j}\right)\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta+\right.\\
&\qquad\left.+\exp\left(-\frac{1+\xi}{2\mu_j}\right)\frac{d}{d\xi}\int_{-1}^{1}\ln\frac{1}{|\xi-\eta|}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right].
\end{aligned}
$$

For the next investigation one needs to evaluate the limit values of the integrals

$L_0(x)=\int_{-1}^{1}\ln\frac{1}{|x-t|}\frac{T_n(t)}{\sqrt{1-t^2}}dt$ and $L_1(x)=\frac{d}{dx}\int_{-1}^{1}\ln\frac{1}{|x-t|}\frac{T_n(t)}{\sqrt{1-t^2}}dt$.

(25)when $x\to-1-0$ and $x\to1+0$. As it shown at Appendix B, for all n the integrals $L_0(x)$ are bounded and when $x\to-1-0$ and when $x\to1+0$ too. For the integrals $L_1(x)$ the next asymptotic formulas were found

$$
L_1(x)\sim\begin{cases}-\dfrac{\pi\sqrt{2}}{2}(-x-1)^{-\frac{1}{2}},\ n=0\\ -\dfrac{\pi\sqrt{2}}{2}(-1)^n(-x-1)^{-\frac{1}{2}}+2\pi n\displaystyle\sum_{j=0}^{n-1}\frac{(n+1)_j(1-n)_j}{\left(3/2\right)_j j!},\ n\geq1\end{cases}\quad\text{when } x\to-1-0
$$

$$L_1(x) \sim \begin{cases} -\dfrac{\pi\sqrt{2}}{2}(x-1)^{-\frac{1}{2}}, n=0 \\ -\dfrac{\pi\sqrt{2}}{2}(x-1)^{-\frac{1}{2}} + 2\pi n, n \geq 1 \end{cases} \quad \text{when } x \to 1+0 . \tag{26}$$

With the help of the obtained asymptotic formulas for the integrals $L_0(x)$ and $L_1(x)$ behaviours , one can calculate the limits in the expressions (24) and write the formulas for the SIF calculation

$$K_{III}^{a_j} = \frac{\sqrt{\pi}}{2\alpha_j\sqrt{R\mu_j}}\sum_{n=0}^{\infty}(-1)^n\theta_{jn},\ K_{III}^{b_j} = \frac{\sqrt{\pi}}{2\beta_j\sqrt{R\mu_j}}\sum_{n=0}^{\infty}\theta_{jn} \tag{27}$$

6. Discussion of the numerical results

The SIF calculations were conducted for a solid and a hollow cylinder with the ratio of radius and height $\gamma = R:H = 1:4$. The ratio of internal radius to the external one for the hollow cylinder is $\varepsilon = R_0 : R = 1:5$. For both problems the load was taken as the uniformly distributed torsion load, applied at the upper part of a lateral surface $q(z) = \begin{cases} P^*, cH < z < H \\ 0.\ 0 < z < cH \end{cases}$, where $p_k = P^*\lambda_k^{-1}\cos\lambda_k c,\ c = 0.8$ and $P^* = 1$.

The results of the SIF calculation for the hollow cylinder $(0 \leq \rho \leq 1)$ for the case of one crack are shown at Table 1 on dependence on crack's size $(\alpha_1;\beta_1)$ and height of its location δ_1 .

Table 1. SIF values for the hollow cylinder $(0 \leq \rho \leq 1)$ for the case of one crack.

	$\delta_1 = 0.25$			$\delta_1 = 0.5$	$\delta_1 = 0.75$	
$(\alpha_1;\beta_1)$	$K_{III}^{a_1}$	$K_{III}^{b_1}$	$K_{III}^{a_1}$	$(\alpha_1;\beta_1)$	$K_{III}^{a_1}$	$K_{III}^{b_1}$
$(0.4;0.6)$	0.8405	0.9383	0.8401	$(0.4;0.6)$	0.8405	0.9383
$(0.3;0.7)$	1.0876	1.4176	1.0871	$(0.3;0.7)$	1.0876	1.4176
$(0.2;0.8)$	1.1784	1.9529	1.1779	$(0.2;0.8)$	1.1784	1.9529
$(0.1;0.9)$	1.1616	2.8761	1.1612	$(0.1;0.9)$	1.1616	2.8761

The results demonstrate that the SIF value depends more significantly on the size of the crack than on its height. For the taken load $K_{III}^{b_1} > K_{III}^{a_1}$ and hence, the increasing of load value will lead to the crack's development at the points of its external contour.

The results of the SIF calculation for the solid cylinder $(0 \leq \rho \leq 1)$ for the case of two cracks are shown in Table 2 on the dependence of the cracks' sizes $(\alpha_i;\beta_i)$ and heights of its location $\delta_i\ (i = 1,2)$.

From these data one concludes that existence of a second crack decreases SIF values of the first crack. At the same time this effect is greater the closer the second crack.

It was interesting to investigate the case when both cracks are situated in one plane SIF values are shown for this case in Table 3. (the change of the kernels of the system of the integral equation (20) corresponding to this case are presented in Application C).

Table 2. SIF values for the solid cylinder $(0 \le \rho \le 1)$ for the case of two cracks.

$(\alpha_1;\beta_1)$ $(\alpha_2;\beta_2)$	$\delta_1 = 0.25; \delta_2 = 0.5$		$\delta_1 = 0.5; \delta_2 = 0.75$		$\delta_1 = 0.25; \delta_2 = 0.75$	
	$K_{III}^{a_i}$	$K_{III}^{b_i}$	$K_{III}^{a_i}$	$K_{III}^{b_i}$	$K_{III}^{a_i}$	$K_{III}^{b_i}$
$(0.3;0.7)$	1.0831	1.4156	1.0829	$(0.3;0.7)$	1.0831	1.4156
$(0.3;0.7)$	1.0826	1.4153	1.0080	$(0.3;0.7)$	1.0826	1.4153
$(0.3;0.7)$	1.0863	1.4170	1.0859	$(0.3;0.7)$	1.0863	1.4170
$(0.4;0.6)$	0.8369	0.9358	0.7805	$(0.4;0.6)$	0.8369	0.9358
$(0.4;0.6)$	0.8373	0.9361	0.8371	$(0.4;0.6)$	0.8373	0.9361
$(0.3;0.7)$	1.0859	1.4168	1.0112	$(0.3;0.7)$	1.0859	1.4168

Table 3. SIF values for the solid cylinder $(0 \le \rho \le 1)$ for the case of two cracks situated on one plane.

$(\alpha_1;\beta_1)$ $(\alpha_2;\beta_2)$	$K_{III}^{a_1}$ $K_{III}^{a_2}$	$K_{III}^{b_1}$ $K_{III}^{b_2}$	$(\alpha_1;\beta_1)$ $(\alpha_2;\beta_2)$	$K_{III}^{a_1}$ $K_{III}^{a_2}$	$K_{III}^{b_1}$ $K_{III}^{b_2}$
$(0.2;0.3)$	0.3033	0.3406	$(0.1;0.5)$	0.4936	0.9112
$(0.5;0.6)$	0.6829	0.7144	$(0.6;0.7)$	0.8851	0.8891
$(0.2;0.3)$	0.3666	0.4195	$(0.1;0.5)$	0.5367	1.0336
$(0.5;0.8)$	1.3982	1.6037	$(0.6;0.8)$	1.3421	1.3939
$(0.2;0.3)$	0.4347	0.5026	$(0.1;0.5)$	0.6178	1.2401
$(0.5;0.9)$	1.8905	2.3537	$(0.6;0.9)$	1.8738	2.0847

As can be seen from the presented results, the SIF value on the external crack is always significantly greater than on the internal crack.

The results of SIF calculation for the hollow cylinder $(0.2 \le \rho \le 1)$ for the case of one crack are shown at Table 4 depending on crack's size $(\alpha_1;\beta_1)$ and the height of its location δ_1.

Table 4. SIF values for the hollow cylinder for the case of one crack.

	$\delta_1 = 0.25$		$\delta_1 = 0.5$		$\delta_1 = 0.75$	
$(\alpha_1;\beta_1)$	$K_{III}^{a_1}$	$K_{III}^{b_1}$	$K_{III}^{a_1}$	$(\alpha_1;\beta_1)$	$K_{III}^{a_1}$	$K_{III}^{b_1}$
$(0.4;0.8)$	1.4337	1.7824	1.4333	$(0.4;0.8)$	1.4337	1.7824
$(0.3;0.9)$	1.8066	2.7711	1.8061	$(0.3;0.9)$	1.8066	2.7711

In comparison with a solid cylinder the value of SIF are greater on the internal contour of the crack, although they are still less than on the external contour of the crack. The absolute values of SIF on the external contour are greater too.

The results of SIF calculation for the hollow cylinder $(0.2 \le \rho \le 1)$ for the case of two cracks are shown in Table 5 depending on the cracks' sizes $(\alpha_i;\beta_i)$ and heights of its location δ_i $(i=1,2)$.

Table 5. SIF values for the hollow cylinder for the case of two cracks.

$(\alpha_1;\beta_1)$ $(\alpha_2;\beta_2)$	$\delta_1=0.25;\delta_2=0.5$		$\delta_1=0.5;\delta_2=0.75$		$\delta_1=0.25;\delta_2=0.75$	
	$K_{III}^{a_i}$	$K_{III}^{b_i}$	$K_{III}^{a_i}$	$K_{III}^{b_i}$	$K_{III}^{a_i}$	$K_{III}^{b_i}$
$(0.4;0.8)$	1.4299	1.7815	1.4297	$(0.4;0.8)$	1.4299	1.7815
$(0.4;0.8)$	1.4295	1.7814	1.3528	$(0.4;0.8)$	1.4295	1.7814
$(0.4;0.8)$	1.4268	1.7808	1.4267	$(0.4;0.8)$	1.4268	1.7808
$(0.3;0.9)$	1.8015	2.7709	1.7137	$(0.3;0.9)$	1.8015	2.7709
$(0.3;0.9)$	1.8020	2.7709	1.8017	$(0.3;0.9)$	1.8020	2.7709
$(0.4;0.8)$	1.4264	1.7808	1.3497	$(0.4;0.8)$	1.4264	1.7808

The exact same trend remains here as it was for the case of the solid cylinder: the existence of a second crack decreases the SIF value of the first crack. Furthermore, as for the case of one crack, SIF values on the internal contour of crack significantly increased in comparison with a solid cylinder.

Conclusions

1. The proposed method allows solution of the torsion problem for the finite elastic multilayered cylinder with ring shaped cracks.

2. The formulas for SIF value calculations are constructed. SIF are investigated depending on the elastic properties of the layers and cracks' location.

3. The proposed method can be used in the case of dynamical torsion load.

Acknowledgements

The research is supported by Ukrainian Department of Science and Education under Project No 0115U003211. The authors wish to express their deep gratitude to Simon Dyke for his invaluable assistance for the text editing.

Appendix A. The singularity of the equation's kernel

Corresponding to the formula (6.576.2 , Gradshtein et al (1963)) one writes

$$W_{00}^{0}(\rho,t)=\int_{0}^{\infty}J_0(\rho x)J_0(tx)\,dx=\frac{1}{\rho+t}F\left(\frac{1}{2},\frac{1}{2};1;\frac{4\rho t}{(\rho+t)^2}\right).$$

Taking into consideration the expression of the full elliptical integral of 1-st order $K(x)$ with the Gauss hypergeometric function (8.113, Gradshtein et al (1963)) one gets finally $W_{00}^{0}(\rho,t)=\dfrac{2}{\pi(\rho+t)}K\left(\dfrac{2\sqrt{\rho t}}{\rho+t}\right)$.

This expression is symmetrical with respect to the ρ and t, so this formula is true as $\rho < t$, and as $t < \rho$ too.

Appendix B. The limit values of the integrals (25)

Let's find the limit values of the integrals (25) when $x \to -1-0$ and $x \to 1+0$. It simpler to provide all reasoning for the integrals of a more common structure with Jacobi polynomials $P_n^{\alpha,\beta}(x)$. Let's use the equality that leads from the formulas (10.8.19 and 10.8.20, Beitman et al (1974))

$$\int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{x-t} P_n^{\alpha,\beta}(t)\,dt = -\pi \frac{(x-1)^{\alpha}(x+1)^{\beta}}{\sin \alpha\pi} P_n^{\alpha,\beta}(x) + \tag{28}$$

$$+2^{\alpha+\beta} \frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}\right), \text{ where } \alpha > -1, \beta > -1,\ x \notin [-1;1]$$

and $F(a,b;c;x)$ Gauss hypergeometric function.

Let's integrate this integral by parts

$$\int_{-1}^{1} \ln\frac{1}{|x-t|}(1-t)^{\alpha}(1+t)^{\beta} P_n^{\alpha,\beta}(t)\,dt = \begin{bmatrix} u = \ln\dfrac{1}{|x-t|} \quad du = \dfrac{dt}{x-t} \\ dv = (1-t)^{\alpha}(1+t)^{\beta} P_n^{\alpha,\beta}(t)\,dt \\ v = -\dfrac{1}{2n}(1-t)^{\alpha+1}(1+t)^{\beta+1} P_{n-1}^{\alpha+1,\beta+1}(t) \end{bmatrix} =$$

$$= -\ln\frac{1}{|x-t|}\cdot\frac{1}{2n}(1-t)^{\alpha+1}(1+t)^{\beta+1} P_{n-1}^{\alpha+1,\beta+1}(t)\Big|_{-1}^{1} + \frac{1}{2n}\int_{-1}^{1}\frac{1}{x-t}(1-t)^{\alpha+1}(1+t)^{\beta+1} P_{n-1}^{\alpha+1,\beta+1}(t)\,dt =$$

$$= -\frac{\pi(x-1)^{\alpha+1}(1+x)^{\beta+1}}{2n\sin(\alpha+1)\pi} P_{n-1}^{\alpha+1,\beta+1}(x) + 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{n\Gamma(n+\alpha+\beta+2)} F\left(n,-n-\alpha-\beta-1;-\alpha;\frac{1-x}{2}\right).$$

It is seen from the result that the initial integral has final limits when $x \to -1-0$ and $x \to 1+0$.

Taking into account that $T_n(x) = \dfrac{\sqrt{\pi}\, n!}{\Gamma\left(n+\frac{1}{2}\right)} P_n^{-\frac{1}{2},-\frac{1}{2}}(x)$, a $U_n(x) = \dfrac{\sqrt{\pi}(n+1)!}{2\Gamma\left(n+\frac{3}{2}\right)} P_n^{\frac{1}{2},\frac{1}{2}}(x)$

one gets

$$L_0(x) = -\frac{\pi}{n}\sqrt{x^2-1}\,U_{n-1}(x) + \frac{\pi}{n} F\left(n,-n;\frac{1}{2};\frac{1-x}{2}\right).$$

This implies that $L_0(x) \to \dfrac{\pi}{n}(-1)^n$ when $x \to -1-0$ and $L_0(x) \to \dfrac{\pi}{n}$ when $x \to 1+0$.

Let's consider the integral

$$\frac{d}{dx}\int_{-1}^{1} \ln\frac{1}{|x-t|}(1-t)^{\alpha}(1+t)^{\beta} P_n^{\alpha,\beta}(t)\,dt = \int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)\beta}{t-x} P_n^{\alpha,\beta}(x)\,dx =$$

$$= \pi\frac{(x-1)^{\alpha}(x+1)^{\beta}}{\sin\alpha\pi} P_n^{\alpha,\beta}(x) - 2^{\alpha+\beta}\frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}\right).$$

Taking into consideration equality (28), one should pass here from Jacobi's polynomials to the Chebyshov's polynomials. It gives

$$L_1(x) = -\pi\frac{T_n(x)}{\sqrt{x^2-1}} + 2\pi n F\left(n+1,1-n;\frac{3}{2};\frac{1-x}{2}\right).$$

This implies that $L_1(x) \sim -\frac{\pi\sqrt{2}}{2}(-1)^n(-x-1)^{-\frac{1}{2}} + 2\pi n\sum_{j=0}^{n-1}\frac{(n+1)_j(1-n)_j}{\left(\frac{3}{2}\right)_j j!}$ when $x \to -1-0$

and $L_1(x) \sim -\frac{\pi\sqrt{2}}{2}(x-1)^{-\frac{1}{2}} + 2\pi n$ when $x \to 1+0$.

Let's consider the special case when $n = 0$. One can use the approach proposed by G. Ya. Popov in Popov (1968). One considers the integral

$$L(z) = \int_{-1}^{1}\ln(z-\zeta)(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}d\zeta,\ z \notin [-1;1].$$

and function $g(z) = (z-1)^{-\frac{1}{2}}(z+1)^{-\frac{1}{2}}$, given on the complex variable z along the cut segment $[-1;1]$. It has value $|g(z)|\exp\left(\frac{i\pi}{2}\right)$ on the upper branch of cut, and value $|g(z)|\exp\left(-\frac{i\pi}{2}\right)$ on the lower branch. On the base of Cauchy formula one can write $g(z) = \frac{1}{2\pi i}\oint_{C^-}\frac{g(\zeta)}{\zeta - z}d\zeta$, where C^- is arbitrary closed contour, covering segment $[-1;1]$, point z is situated outside contour C^-. We contract the contour to the segment $[-1;1]$. As a result one gets

$$(z-1)^{-\frac{1}{2}}(z+1)^{-\frac{1}{2}} = -\frac{1}{\pi}\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\frac{d\zeta}{\zeta - z},\ z \notin [-1;1]. \tag{29}$$

Let's integrate the equality (29) along some curve connecting point $z = 1$ with arbitrary point z:

$$\int_1^z (s-1)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}ds = -\frac{1}{\pi}\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}d\zeta\int_1^z\frac{ds}{\zeta - s}. \tag{30}$$

The variable change $s = \frac{1+u\tau}{1-u\tau}$ where $u = \frac{z-1}{z+1}$ is done at the left hand part of the equality

$$\int_1^z (s-1)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}ds = u^{\frac{1}{2}}\int_0^1\tau^{-\frac{1}{2}}(1-u\tau)^{-1}d\tau = 2u^{\frac{1}{2}}F\left(1,\frac{1}{2};\frac{3}{2};u\right) = 2\sqrt{\frac{z-1}{z+1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{z-1}{z+1}\right),$$

with regard to the integral expression (9.111, Gradshtein et al (1963)) for Gauss hypergeometric function.

Let's consider the right hand part of the equality (30). With the known expression $\int_1^z\frac{ds}{\zeta - s} = \ln(1-\zeta) - \ln(z-\zeta)$, one gets

$$\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}d\zeta\int_1^z\frac{ds}{\zeta - s} = \int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(1-\zeta)d\zeta - L(z).$$

In the first integral on the right hand side the change of variable is provided $\zeta = 1-2t$:

$$\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(1-\zeta)d\zeta=\int_{0}^{1}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\ln t\,dt+\ln 2\int_{0}^{1}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right],$$

where $\psi(x)$ Euler Phi function. Thus, it is found

$$L(z)=2\pi\sqrt{\frac{z-1}{z+1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{z-1}{z+1}\right)+\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right].$$

Let’s take here $z=x>1$. Then

$$\int_{-1}^{1}\ln(x-t)\frac{dt}{\sqrt{1-t^2}}=2\pi\sqrt{\frac{x-1}{x+1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{x-1}{x+1}\right)+\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right] \tag{31}$$

and it leads from the obtained equality that for $n=0$

$L_0(x)\to-\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right]$ when $x\to 1+0$. It can be seen there is no singularity here.

After the differentiation of (31) one gets

$$L_1(x)=-2\pi\left[(x+1)^{-2}\sqrt{\frac{x+1}{x-1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{x-1}{x+1}\right)+\frac{2}{3}(x+1)^{-2}\sqrt{\frac{x-1}{x+1}}F\left(2,\frac{3}{2};\frac{5}{2};\frac{x-1}{x+1}\right)\right].$$

Hence finally for $n=0$ $L_1(x)\sim-\frac{\pi\sqrt{2}}{2}(x-1)^{-\frac{1}{2}}$, if $x\to 1+0$.

Let’s consider the case when $x\to-1-0$. The equality (29) should be integrated along some curve connecting point $z=-1$ with arbitrary point z

$$\int_{-1}^{z}(s-1)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}ds=-\frac{1}{\pi}\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}d\zeta\int_{-1}^{z}\frac{ds}{\zeta-s}.$$

Integral in left hand part with the change of variable $s=-\frac{1+u\,\tau}{1-u\,\tau}$, where $u=\frac{z+1}{z-1}$, is calculated by the analogous way

$$\int_{-1}^{z}(s-1)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}ds=-2\sqrt{\frac{z+1}{z-1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{z+1}{z-1}\right).$$

With the fact that $\int_{-1}^{z}\frac{ds}{\zeta-s}=\ln(1+\zeta)-\ln(\zeta-z)$, the integral in right hand part takes the form

$$\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}d\zeta\int_{-1}^{z}\frac{ds}{\zeta-s}=\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(1+\zeta)d\zeta-\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(\zeta-z)d\zeta.$$

Let’s change variable $\zeta=2t-1$ of the first integral in the right hand part. Then

$$\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(1+\zeta)\,d\zeta=\int_{0}^{1}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\ln t\,dt+\ln 2\int_{0}^{1}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt=\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right].$$

Hence

$$\int_{-1}^{1}(1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}\ln(\zeta-z)\,d\zeta=-2\pi\sqrt{\frac{z+1}{z-1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{z+1}{z-1}\right)+\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right].$$

When $z=x<-1$

$$\int_{-1}^{1}\ln(t-x)\frac{dt}{\sqrt{1-t^2}}=-2\pi\sqrt{\frac{x+1}{x-1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{x+1}{x-1}\right)+\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right]. \tag{32}$$

It leads from this constructed equality that for $n=0$

$L_0(x)\to-\pi\left[\ln 2+\psi\left(\frac{1}{2}\right)-\psi(1)\right]$ when $x\to-1-0$, there is no singularity here too·

After the differentiation of constructed equality (32)

$$L_1(x)=-2\pi\left[(x-1)^{-2}\sqrt{\frac{x-1}{x+1}}F\left(1,\frac{1}{2};\frac{3}{2};\frac{x+1}{x-1}\right)+\frac{2}{3}(x-1)^{-2}\sqrt{\frac{x+1}{x-1}}F\left(2,\frac{3}{2};\frac{5}{2};\frac{x+1}{x-1}\right)\right].$$

Hence, for $n=0$ $L_1(x)\sim-\frac{\pi\sqrt{2}}{2}(-x-1)^{-\frac{1}{2}}$, when $x\to-1-0$.

Finally we show that for all n integrals $L_0(x)$ are bounded when $x\to1+0$, and when $x\to-1-0$. The asymptotic formula (26) is constructed for integrals $L_1(x)$

Appendix C.

Suppose for definiteness that two cracks r-d and l-d are located in one plane $\delta_r=\delta_l=\delta.$ Then in the system of the integral equations (17) only formulas for the kernel's are changed

$$S_{rl}(\rho,t)=S_{lr}(\rho,t)=\int_0^\infty J_1(\rho x)J_1(tx)\left[sh\frac{x}{\gamma}(1-2\delta)+sh\frac{x}{\gamma}\right]x^2\sec h\frac{x}{\gamma}dx.$$

The corresponding kernels of system (18) will be changed

$$S^*_{rl}(\rho,t)=S^*_{lr}(\rho,t)=\int_0^\infty J_0(\rho x)J_0(tx)\left[sh\frac{x}{\gamma}(1-2\delta)+sh\frac{x}{\gamma}\right]\sec h\frac{x}{\gamma}dx.$$

After extraction of the weakly convergent part one gets

$$S^*_{rr}(\rho,t)=W^0_{00}(\rho,t)+S^*_{rr}(\rho,t),\quad S^*_{ll}(\rho,t)=W^0_{00}(\rho,t)+S^*_{ll}(\rho,t).$$

Summands $W^0_{00}(\rho,t)$ haven't singularities, because variables ρ and t are changed here on the nonintersecting intervals $(\alpha_r;\beta_r)$ and $(\alpha_l;\beta_l)$. For their calculation one can use formula (1.12.31.1, Prudnikov et al (1983))

$$W_{00}^{0}\left(\rho,t\right)=\frac{2}{\pi}\begin{cases}t^{-1}K\left(\dfrac{\rho}{t}\right), 0<\rho<t\\ \rho^{-1}K\left(\dfrac{t}{\rho}\right), 0<t<\rho\end{cases}$$

In accordance with the above the kernels $M_{rl}\left(\xi,\eta\right)$ and $M_{lr}\left(\xi,\eta\right)$ of integral equations system (20) and coefficients B_{mn}^{rl} and B_{mn}^{lr} of linear algebraic equations infinite system (22) will be changed too.

References

Akiyawa, T., Hara, T., Shibua, T., 2001. Torsion of an infinite Cylinder with Multiple Parallel Circular Cracks. Theor. Appl. Mech. 50, 143–173.

Aleksandrov, V.M., Smetanin, B.I., Sobol, B.V., 1993. Thin concentrator of stresses in elastic bodies (in Russian). Fizmatgiz, Moscow, pp. 224.

Babeshko, V.A., Ratner, S.V., Syromyatnikov, P.V., 2007. Anisotropic bodies with inhomogeneities: The case of a set of cracks. Mechanics of Solids 42, 5, 700-709.

Beitman, G., Erderi, A., 1974. Higher transcendental functions (in Russian), 2. Nauka, Moscow, pp. 296.

Chang, D., Kotousov, A.A., 2014. A fatigue crack growth model for interacting cracks in a plate of arbitrary thickness. Fatigue & Fracture of Engineering Materials & Structures 37, 11, 1254–1267.

Chang, S.S., 1985. The general solution of a finite cylinder with a concentric penny-shaped crack under torsion. Engineering Fracture Mechanics 22, 4, 571-578.

Gradshtein, L., Rygik, L., 1963. The tables of integrals, series and products (in Russian). Nauka, Moscow.

Gribova, M.V., Nuller, B.M., 1989. Torsion of a circular cylinder weakened by a cylindrical crack. Soviet Applied Mechanicks 25, 3, 222-225.

Hakobyan, V., 2014. Mixed Boundary Value Problems On interaction of continuum deformable bodies with the different types stress concentrators (in Russian). Gitutyan, Yerevan.

Han Xue-Li, Wang Duo, 1994. A circular or ring-shaped crack in a nonhomogeneous cylinder under torsional loading. International Journal of Fracture 68, 3, R79-R83.

Huang, G.-Y., Wang, Y.-S., Yu, S.-W., 2005. Stress concentration at a penny-shaped crack in a nonhomogeneous medium under torsion. Acta Mechanica 180, 107–115.

Jin-Chad Yue, Ren-Ji Tang, 1996. Integral equation method for the torsion of a composite cylinder with crack and inclusion. Engineering Fracture Mechanics 55, 5, 763-775.

Kaman, M.O., Mehmet, R.G., 2006. Cracked semi-infinite cylinder and finite cylinder problems. Int. J. Eng. Sci. 44, 20, 1534–1555.

Kamke, E., 1976. Handbook of ordinary differential equations (in Russian). Nauka, Moscow, pp. 576.

Kanwal, R.P., Pasha, M.I., 1974. Axially Symmetric Stress Distributions in Elastic Solids Containing Ring-Shaped Cracks Under Torsion. J. Appl. Mech. 41, 2, 516-517.

Kudryavcev, B.I., Parton, V.Z., 1973. Torsion and stretching of a cylinder with an outer annular slit (in Russian). Applied mathematics and mechanics 37, 2, 316-325.

Lee, D-S., 2004. The problem of internal cracks in an infinite strip having a circular hole. Acta Mechanica 169, 101-110.

Malits, P., 2009. Torsion of a cylinder with a shallow external crack. Int. J. of Solids and Structures 46, 3061–3067.

Morozov, N.F., 1984. Mathematical questions of the crack's theory (in Russian). Nauka, Moskow.

Mykhas'kiv, V., Stankevych, V., Zhbadynskyi, I, Zhang, Ch., 2009. 3-D dynamic interaction between a penny-shaped crack and a thin interlayer joining two elastic half-spaces. International Journal of Fracture 159, 2, 137-149.

Panasyuk, V.V., Andrejkiv, A.E., Stadnik, M.M., 1981. Three-dimensional static crack problems solution (a review). Engineering Fracture Mechanics 14, 2, 245-260.

Pengpeng Shi, Xiaojing Zheng, 2015. The Yoffe-type moving tubular interface crack in a hollow composite cylinder with finite length. International Journal of Mechanical Sciences 98, 29-38.

Popov, G.Ya., 1968. On one remarkable property of the Jacobi polynomials (in Russian). Ukr. mathem. journal 20, 4, 540-547.

Popov, G.Ya., 1982. The elastic stress' concentration around dies, cuts, thin inclusions and reinforcements (in Russian). Nauka, Moskow.

Protserov, Yu., Vaysfeld, N., 2017. The torsion problem for the elastic multilayered finite cylinder with the circular crack.Applied Mathematics and Mechanics (English Edition)

Prudnikov, A.P., Brichkov, Yu.A, Marichev, O.I., 1983. Integrals and series. Special functions (in Russian). Nauka, Moscow, pp. 752.

Savruk, M.P., Osiv, P.N., Prokopchuk, I.V., 1989. Numerical analysis in plane problems of the crack's theory (in Russian). Naukova dumka, Kyiv.

Shi Peng Peng, 2015. The tubular interface crack in a hollow composite cylinder under static torsion with four frequently encountered constraint edges. Journal of Mechanical Science and Technology 25, 9, 3805-3818.

Suzuki, K., Shibuya, T., Koizumi, T., 1980. The torsion of hollow cylinder with two or infinite parallel external circular cracks. Bull. JSME 23, 179, 637–643.

Wuthrich, C., 1980. Stress intensity factors for cylindrical cracks in a long cylinders. Eng. Fract. Mech. 13, 4, 987-990.

Xie, Y.J., Wang, X.H., 2003. Crack initiation and direction for circumferential periodic cracks in pipe under tension and torsion. Theoretical and Applied Fracture Mechanics 40, 2, 153–159.

Yantian Lin, Hong Yuan, 1988. Torsion of a finite length laminar composite circular cylinder deboned over a penny shaped area. Eng. Fracture Mechanics 31, 1, 1–8.
Yong Dong Li, Hao Zha, Tao Xiong, 2013. The cylindrical interface crack in a layered tubular composite of finite thickness under torsion. European Journal of Mechanics A/Solids 39, 113-119.
Zhang, X.S., 1988. Off-plane concentric penny-shaped crack in a finite cylinder under arbitrary torsion. Theoretical and Applied Fracture Mechanics 9, 3, 263-270.